On the Vision of William P. Thurston: Topology has Geometry and Dynamics has Both

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2 Overview

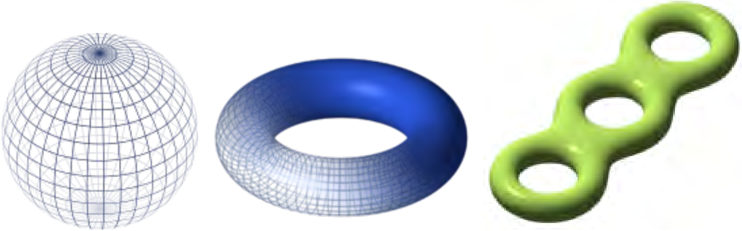
- topology has geometry in dimension 2: surfaces
- topology has geometry for 3-manifolds
- surface homeomorphisms have geometry
- topology and geometry of rational maps on the Riemann sphere
- related proofs: iteration in Teichmüller space
- classification results in complex dynamics: Newton method
- current work: extension of theory to new settings, especially in transcendental world
- the role of discrete models ...
- ... and of intelligent discretization methods

3 Surfaces and their geometry

Major result of 19th century mathematics

Theorem 1 (Topology of surfaces)

Every closed orientable surface is homeomorphic to a sphere with g handles (topology of closed surfaces is completely classified by genus $g \in \mathbb{N}$).



Surfaces of genus $g = 0$ (sphere), $g = 1$ (torus), $g = 3$

Theorem 2 (Geometry of surfaces)

Every closed surface of genus

- $g = 0$ *is homeomorphic to the standard sphere with its spherical geometry*
- $g = 1$ *is homeomorphic to a torus with its Euclidean geometry*
- $g > 1$ *is homeomorphic to a surface with hyperbolic geometry.*

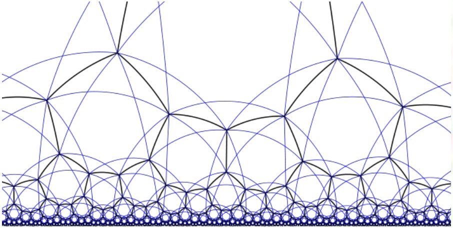
All these surfaces have constant curvature $K > 0$ *(sphere),* $K = 0$ *(torus),* $K < 0$ *(all others): same at all points.*

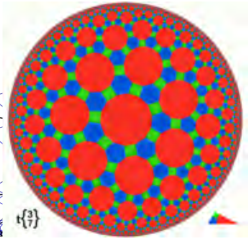
Note: for every $g \geq 1$, this (intrinsic) curvature is *not* equal to the curvature coming from the embedding into $\mathbb{R}^3$!
For fixed $g \geq 1$, the surfaces are not necessarily isometric.

5 What is hyperbolic geometry?

In dimension $n \geq 2$, have *upper half space* $\mathbb{H}_n := \{(x_1, \ldots, x_n) \colon x_n > 0\}$ with infinitesimal metric $ds := \|dx\|/x_n$ (Poincaré upper half space model).

Isometric model: Poincaré ball model: $\mathbb{D}_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon \|x\| < 1\}$ with infinitesimal metric $ds = 2\|dx\|/(1 - \|x\|^2)$.





Conclusion in dimension 2: every domain $U \subset \overline{\mathbb{C}}$ has geometry: if $U = \overline{\mathbb{C}}$ then U is spherical, if $|\overline{\mathbb{C}} \setminus U| \leq 2$ then U conformally equivalent to $\mathbb{C}$ or $\mathbb{C}^*$ and Euclidean, and otherwise hyperbolic (Riemann mapping theorem / uniformization theorem).

6 The hyperbolic disk and Escher tilings

A tiling of the hyperbolic plane by squares and equilateral triangles in the work of M. C. Escher: at every vertex there are 3 squares and 3 triangles!

7 Thurston's program: geometrization conjecture I

Major theorem of 20th century mathematics:

Conjecture (Thurston's vision on 3-manifolds, 1980's)

*Every closed oriented 3-manifold is geometric.
More precisely, every closed oriented 3-manifold can be decomposed in a canonical way into pieces so that each piece carries one of eight standard geometric structures.*

The *decomposition* is classical (Kneser, Milnor, Jaco-Shalen): first, every closed 3-manifold can be decomposed into *prime* manifolds (essentially equivalent to *irreducible:* every embedded 2-sphere bounds a 3-ball).

Thurston conjecture: Every closed oriented prime 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with finite volume.

Special case of geometrization conjecture: Poincaré conjecture. Much of it proved by Thurston, in generality by Perelman.

8 Thurston's program: geometrization conjecture II

Conjecture (Thurston's vision on 3-manifolds, 1980's)

Every closed oriented 3-manifold can be decomposed in a canonical way into pieces so that each piece carries one of eight standard geometric structures.

Thurston gave a classification of all relevant geometries:

- spherical $\mathbb{S}^3$, Euclidean $\mathbb{R}^3$, hyperbolic $\mathbb{H}^3$,
- two products: $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$,
- three special geometries: Sol, Nil, $\widetilde{SL_2(\mathbb{R})}$

By far the most frequent of these is hyperbolic geometry, and Thurston proved several hyperbolization theorems (for instance on knot complements in 3-sphere).

Theorem 3 (Geometry of surface automorphisms)

Let S be a closed surface and $f\colon S \to S$ be a homeomorphism. Then up to homotopy f has geometry (pseudo-Anosov) unless f is periodic or reducible ("geometry or obstruction").

The *pseudo-Anosov geometry of f* means the following: there are two transverse foliations of S by lines with singularities at finitely many points so that f is a K-stretch (with $K > 1$) along the first foliation, and a K-contraction along the second one.

The map f *is periodic* if there is an $n \geq 1$ so that $f^{\circ n}$ is homotopic to the identity.

The map is *reducible* if there are finitely many disjoint essential (=non-contractible) simple closed curves that are permuted by f, up to homotopy.

10 Thurston's hyperbolization theorems

Theorem 4 (Mapping torus is hyperbolic)

Let S be a closed surface, $f\colon S \to S$ be a surface automorphism. Define the mapping torus $M_f := S \times [0,1]/(p,0) \sim (f(p),1)$.
Then M_f is hyperbolic if and only if f is pseudo-Anosov.

This uses the classification of surface automorphisms.
Pseudo-Anosov means that $f\colon S \to S$ has geometry.

Theorem 5 (Manifolds that do not fiber over the circle)

Every closed irreducible atoroidal Haken 3-manifold is hyperbolic.

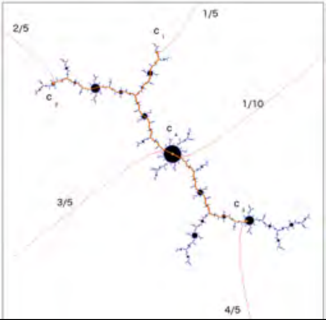
Haken manifold is "sufficiently big": contains essential embedded surface S (fundamental group of S injects into fundamental group of 3-manifold). *Atoroidal:* no essential torus (along such tori further decomposition necessary).

11 Characterization of rational maps

Thurston mapping: a finite degree branched cover $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ with finitely many branch points, and all branch points have finite orbits (are periodic or preperiodic)

Theorem 6 (Characterization of rational maps)

Every Thurston mapping is either “realized” by a rational map (i.e., has invariant complex structure: “has geometry”) or it has a “multicurve obstruction”. — “Geometry or obstruction”



Example of a map $z \mapsto p_c(z) = z^2 + c$ where the critical point $z = 0$ has period 4: $0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c) + c \mapsto ((c^2+c)^2+c)^2+c = 0$: algebra admits 8 solutions; how to distinguish them?

Answer: classification by "Hubbard trees" (up to homotopy)

12 Teichmüller space and iteration

Definition 7 (Teichmüller space)

For a closed (topological) surface S, perhaps with finitely many punctures, the *Teichmüller space Teich*(S) is the set of pairs (X, ϕ), where X is a Riemann surface homeomorphic to S and $\phi\colon S \to X$ is a homeomorphism, subject to the equivalence relation $(X_1, \phi_1) \sim (X_2, \phi_2)$ if there exists a conformal isomorphism $h\colon X_1 \to X_2$ so that $\phi_2 = h \circ \phi_1$ *up to homotopy*.

All three theorem classes (surface automorphisms, hyperbolization of 3-manifolds, characterization of rational maps) are proved by iteration in a finite-dimensional Teichmüller space: a fixed point (or a point with minimal distance to image) yields geometry, otherwise get combinatorial obstruction in form of invariant multicurve.

A simple closed curve $\gamma \subset S$ is *essential* if it cannot be contracted to a point; a *multicurve* is a finite collection of disjoint, non-homotopic essential simple closed curves.

13 Surface automorphisms

Given surface S, have automorphism $f\colon S \to S$.

Teichmüller space of S consists of pairs (X, ϕ) where X is a Riemann surface and $\phi\colon S \to X$ is a homeomorphism.

Let $\delta := \inf(d((X, \phi), (X, \phi \circ f))$ (distance in Teichmüller space);

a) if $\delta = 0$ and infimum realized (i.e, there exists a fixed point in Teichmüller space), then f is periodic;

b) if $\delta > 0$ and infimum realized (i.e., there exists a point with minimal distance to image), then have extremal geometric structure on S, yields pseudo-Anosov structure ("geodesic in Teichmüller space");

c) in either case, if infimum *not* realized, then obtain invariant multicurve on S: f is *reducible*. (Can decompose S along the multicurve, obtain surface with boundary punctures, continue).

Details: John Hubbard, *Teichmüller theory*, vol. 2 (2016).

14 Hyperbolization of 3-manifolds

Key step in proof: glue two hyperbolic manifolds with boundary (X_1, S_1) and (X_2, S_2) along their boundary by a gluing map $\phi\colon S_1 \to S_2$, want it so that the gluing respects hyperbolic structure so that the gluing result $X_1 \cup_\phi X_2$ becomes hyperbolic.

This “boundary surface matching” involves iteration on Teichmüller space of S_1;

a) fixed point in $Teich(S_1)$ yields compatible geometric structure;

b) non-existence of fixed point yields invariant multicurve in S_1; extends through X into invariant torus: so manifold is not atoroidal.

Details: John Hubbard, *Teichmüller theory*, vol. 3/4 (forthcoming).

15 Characterization of rational maps

Thurston mapping: a branched cover $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ with finitely many branch points, and all branch points have finite orbits (are periodic or preperiodic).

Theorem 8 (Characterization of rational maps)

Every Thurston mapping is either "realized" by a rational map (i.e., has invariant complex structure: "has geometry") or it has a "multicurve obstruction". — "Geometry or obstruction"

Let P_f be the finite set of orbits of branch points of f and $S := (\mathbb{S}^2, P_f)$: sphere with finitely many marked points. Then f induces iteration on Teich(S): given $\phi_0\colon S \to (\overline{\mathbb{C}}, \phi_n(P))$, then

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\ \phi_1\ } & (\overline{\mathbb{C}}, P_g) \\ \downarrow f & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{\ \phi_0\ } & (\overline{\mathbb{C}}, P_g) \end{array}$$

by pull-back of complex structures and the uniformization theorem. Then $\phi_0 \sim \phi_1$ iff $\exists$ fixed point in Teich iff f is equivalent to rational map g.

16 What can Thurston's "rational map theorem" do?

⋆ Gives explicit form of "multicurve obstruction", but complicated in practice (f-invariant for which associated matrix has leading eigenvalue greater at least 1)

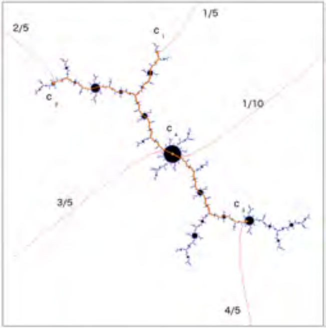
⋆ Can be interpreted in terms of moduli of annuli (so necessity of condition easy to see)

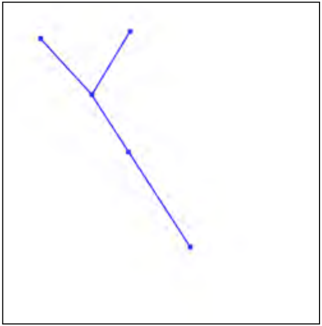
⋆ Serious current activity on checking whether Thurston criterion is satisfied (including Dylan Thurston)

⋆ Every application of Thurston's theorem is a major theorem in its own right:
Step 1: *extract from rational map some combinatorial invariant (tree, graph, lamination, etc)*
Step 2: *describe the resulting invariants*
Step 3: *show that each invariant extends to branched cover of* $\mathbb{S}^2$
Step 4: *show that this cover does not have a Thurston obstruction, so is realized by a rational map.*

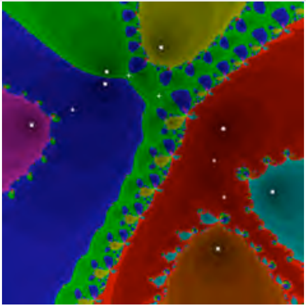
17 Three typical uses of Thurston's theorem

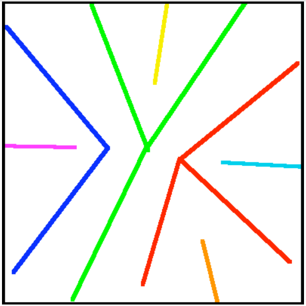
⋆ Classification of postcritically finite polynomials in terms of Hubbard trees and orbit portraits (Poirier 1990's, based on Bielefeld, Fisher, Hubbard).





⋆ Classification of postcritically finite Newton maps in terms of extended Newton trees (Lodge, Mikulich, Schleicher 2015).



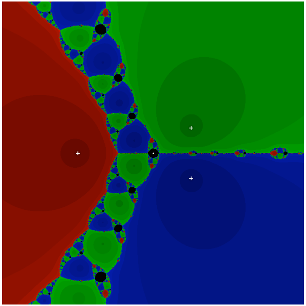


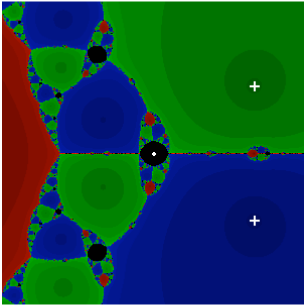
For polynomial p, the associated Newton map is $N_p(z) = z - p(z)/p'(z)$: a rational map usually of same degree. Colors describe basins of attraction of different roots.

Rough classification by *channel diagram*: arcs connecting roots to ∞ through basins. Not enough to distinguish different dynamics: for instance, how do all the little bounded components of basins connect to each other?

(They are all attached to each other; proof 2006/2016).

19 Newton maps with attracting cycles





Observation: there are Newton maps that have attracting cycles of periods 2 or higher: open sets of starting points that fail to converge to roots.

Question (Smale, 1970's): give a classification of Newton maps with this "strange" property.

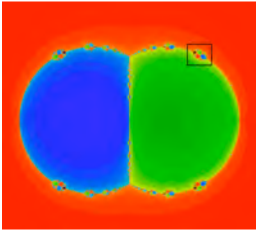
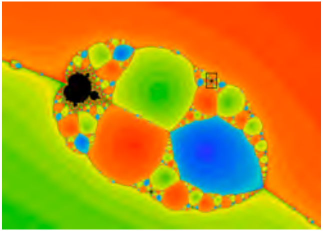
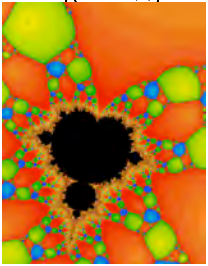
Lodge, Mikulich, Schleicher 2015: classification of all Newton maps (postcritically finite). First "large" space of rational maps with complete classification.

Simplest non-trivial case of Newton maps:
$p(z) = c(z - \lambda)(z - \mu)(z - \nu)$.

Newton dynamics $N_p(z) = z - p(z)/p'(z)$ ignores c.

Translation: may assume $\nu = 0$.

Scaling: may assume $\mu = 1$: hence $p(z) = z(z - 1)(z - \lambda)$.

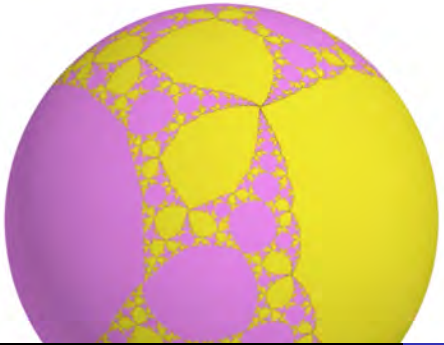




Classification of cubic Newton maps would involve classification of all colored components: black ones are those with attracting cycles. (Serious work by Tan Lei, Roesch, etc).

Space of degree d polynomials has complex dimension $d - 2$.

21 Third example: mating of rational maps

Dynamics of polynomials are (relatively) easy to understand; but how about non-polynomial rational maps?



And even dendrite Julia sets can be mated...!

22 Current work I: transcendental Thurston theorem

Project (with graduate student Bayani Hazemach)

Extend Thurston's theorem from (postcritically finite) rational maps to (postsingularly finite) transcendental maps.

Transcendental maps are postsingularly finite: leads to iteration in *finite-dimensional* Teichmüller space.

Initial work jointly with Hubbard and Shishikura for special case of exponential maps (2009). Thurston's proof has many "finite degree hence finite choice" arguments, these fail for infinite degree. Fresh approach required.

Should lead to dynamical classification of postsingularly finite transcendental entire functions. First results by former undergrads Bastian Laubner and Vlad Vicol (now Princeton).

What are Hubbard trees for transcendental maps? (They don't exist, but what do they look like? What are their properties?)

23 Current work II: Thurston's theorem and infinite-dimensional Teichmüller theory

Project (with grad student Kostya Bogdanov and postdoc Russell Lodge)

Extend Thurston's theorem from postsingularly finite to some ***postsingularly infinite*** *transcendental entire functions.*

"Simple" example: classify all maps $z \mapsto \lambda e^z$, or all maps $z \mapsto \lambda \cos z + \mu \sin z$, for which all singular values converge to ∞ under iteration.

Conceptual context: *dynamic rays* for transcendental entire functions were introduced by Rottenfußer, Rückert, Rempe, Schleicher, *Annals of Math* (2011).

Goal: describe *parameter rays* (external rays in parameter space) for this class of transcendental maps.

Problem: this requires iteration in *infinite*-dimensional Teichmüller space.

24 Conclusion / perspectives

"Topology has geometry" in several incarnations:

- every surface homeomorphism $f\colon S \to S$ has geometry or a multicurve obstruction;
- every 3-manifold can be decomposed into pieces with geometry; most important case: hyperbolic geometry
- important step: gluing hyperbolic 3-manifolds with "topologically compatible" boundaries either is possible in compatible way: yields hyperbolic geometry on union; or multicurve obstruction on boundaries that leads to essential embedded tori;
- every topological postcritically finite branched cover of finite degree has geometry or a multicurve obstruction
- the last result is "fundamental theorem of complex dynamics" for rational maps, has potential to extend to transcendental dynamics, and to extend from finite to infinite dimensional Teichmüller spaces.
- *how do these extended results carry over to other settings such as surface automorphisms and 3-manifolds?*